MAGIC-TDAS 02-03
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Image parameters

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Abstract

Contents

1 Introduction

In experiments using the imaging Cherenkov technique the air showers, induced by cosmic ray particles in the atmosphere, are projected onto the camera. After proper calibration and pedestal subtraction the shower image is represented as the number of Cherenkov photons N_i (and its error ΔN_i) for each pixel i $(i = 1, ...N_{pix})$. For further analysis it is useful to characterize the image by simple parameters. A possible set of parameters are moments of the 2-dimensional distribution of N_i, which were first proposed by Hillas [1, 2, 3, 4].

In the subsequent sections the definition of the image parameters is given, and the formulas for calculating their errors are compiled.

2 Moments of the shower image

Notation (see Fig. 1) :

$$\begin{array}{ll} (x_i, y_i) & \text{coordinates of pixel } i \text{ in original camera system} \\ (x_0, y_0) & \text{reference point (camera center, source position, anti} - \text{source position, ...)} \\ N_i & \text{number of photons in pixel } i \\ w_i = \dfrac{N_i}{\sum_k N_k} & \text{weight for pixel } i \end{array} \tag{1}$$

Definition of moments :

$$\begin{array}{lll} \overline{x} = \sum_i w_i \cdot x_i & \overline{y} = \sum_i w_i \cdot y_i & \\ \overline{x^2} = \sum_i w_i \cdot x_i^2 & \overline{y^2} = \sum_i w_i \cdot y_i^2 & \overline{xy} = \sum_i w_i \cdot x_i \cdot y_i \\ \overline{x^3} = \sum_i w_i \cdot x_i^3 & \overline{y^3} = \sum_i w_i \cdot y_i^3 & \\ \overline{xy^2} = \sum_i w_i \cdot x_i y_i^2 & \overline{x^2y} = \sum_i w_i \cdot x_i^2 y_i & \end{array} \tag{2}$$

Definition of correlations :

$$c_{xx} = \overline{(x - \overline{x})^2} = \overline{x^2} - \overline{x}^2 \qquad c_{yy} = \overline{(y - \overline{y})^2} = \overline{y^2} - \overline{y}^2 \tag{3}$$

$$c_{xy} = \overline{(x - \overline{x})(y - \overline{y})} = \overline{xy} - \overline{x} \cdot \overline{y} \tag{4}$$

$$c_{x^3} = \overline{(x - \overline{x})^3} = \overline{x^3} - 3\overline{x^2} \cdot \overline{x} + 2\overline{x}^3 \qquad c_{y^3} = \overline{(y - \overline{y})^3} = \overline{y^3} - 3\overline{y^2} \cdot \overline{y} + 2\overline{y}^3 \tag{5}$$

$$c_{xy^2} = \overline{(x - \overline{x})(y - \overline{y})^2} = \overline{xy^2} - \overline{x} \cdot \overline{y^2} - 2\overline{xy} \cdot \overline{y} + 2\overline{x} \cdot \overline{y}^2 \tag{6}$$

$$c_{x^2y} = \overline{(x - \overline{x})^2(y - \overline{y})} = \overline{x^2y} - \overline{x^2} \cdot \overline{y} - 2\overline{xy} \cdot \overline{x} + 2\overline{x}^2 \cdot \overline{y} \tag{7}$$

The 2-dimensional correlation matrix C_{orig} in the original camera system reads

$$C_{orig} = \begin{pmatrix} c_{xx} & c_{xy} \\ c_{xy} & c_{yy} \end{pmatrix} \tag{8}$$

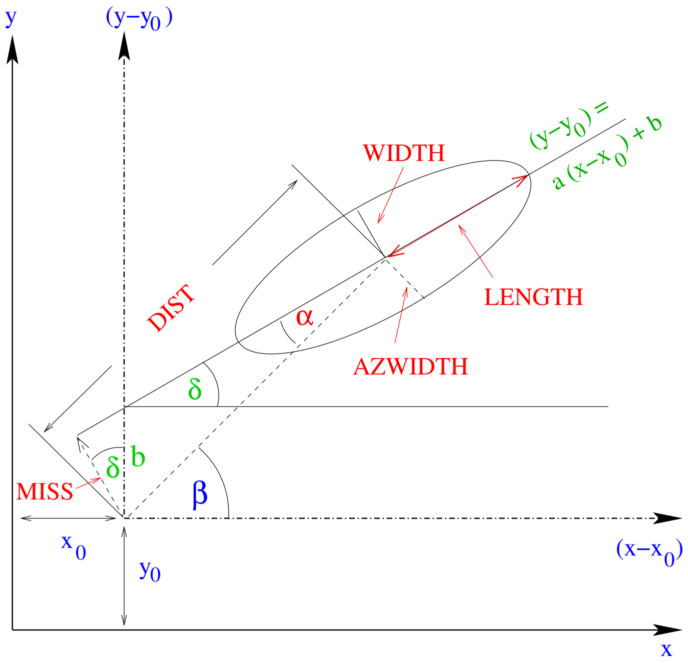

Figure 1: Definition of the image parameters. (x, y) are coordinates in the original camera system. (x_0, y_0) is a reference point, for example the source position or the center of the camera.

C_{orig} and all correlations are invariant against translations, i.e. independent of the position of the shower image in the camera, and therefore independent of the reference point (x_0, y_0).
Introducing the vector notation

$$u = \begin{pmatrix} x \\ y \end{pmatrix} \tag{9}$$

C_{orig} may be written as

$$C_{orig} = \begin{pmatrix} c_{xx} & c_{xy} \\ c_{xy} & c_{yy} \end{pmatrix} = \overline{(u-\overline{u})(u-\overline{u})^T} = \overline{uu^T} - \overline{u}\,\overline{u}^T \tag{10}$$

3 Diagonalisation of the correlation matrix

A new coordinate system is introduced by rotating the original system by an angle δ in the anti-clockwise direction (see Fig. 1). The coordinates in the new system are

$$v = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = Au \tag{11}$$

The correlation matrix $C_{rotated}$ in the rotated system becomes

$$C_{rotated} = \begin{pmatrix} c_{x'x'} & c_{x'y'} \\ c_{x'y'} & c_{y'y'} \end{pmatrix} = \overline{(v-\overline{v})(v-\overline{v})^T} = \overline{vv^T} - \overline{v}\,\overline{v}^T \tag{12}$$

$$= A(\overline{uu^T} - \overline{u}\,\overline{u}^T)A^T = A\overline{(u-\overline{u})(u-\overline{u})^T}A^T \tag{13}$$

$$= A \cdot C_{orig} \cdot A^T \tag{14}$$

The rotation may be chosen such that the correlation matrix is diagonalized ($c_{x'y'} = 0$). The condition

$$c_{x'y'} = \begin{pmatrix} \cos\delta & \sin\delta \end{pmatrix} \begin{pmatrix} c_{xx} & c_{xy} \\ c_{xy} & c_{yy} \end{pmatrix} \begin{pmatrix} -\sin\delta \\ \cos\delta \end{pmatrix} \tag{15}$$

$$= -\frac{1}{2}\sin(2\delta)\cdot(c_{xx} - c_{yy}) + \cos(2\delta)\cdot c_{xy} = 0 \tag{16}$$

yields

$$\tan(2\delta) = \frac{2c_{xy}}{c_{xx} - c_{yy}} = \frac{2\tan\delta}{1 - \tan^2\delta} \tag{17}$$

Setting $d = c_{yy} - c_{xx}$ one gets

$$\tan\delta = \frac{d \pm \sqrt{d^2 + 4c_{xy}^2}}{2c_{xy}} \tag{18}$$

The two signs in the expression for $\tan\delta$ correspond to the ambiguity $\pm 90°$ of δ. Without loss of generality it is sufficient to restrict oneself to the + sign. As will be shown below, δ then describes the rotation into a system where the new x-axis is along the main axis of the image.

The other ambiguity $\pm 180°$ in δ comes from the transformation of $\tan\delta$ into δ. It is sufficient to consider only rotation angles $-90° < \delta < 90°$, because at this point it is not yet specified which end of the image is the head and which one is the tail of the image. The head (tail) of the image corresponds to that part of the shower which is higher (lower) in the atmosphere. Any head-tail information can be taken into account in the definition of the image parameter ALPHA (see below).

Again, since the rotation angle only involves correlations, δ is invariant against translations, i.e. independent of the position of the shower image in the camera.

With the conventions

- positive sign in eq.(18)
- $-90° < \delta < 90°$

all ambiguities are removed and one obtains

$$\tan\delta = \frac{d + \sqrt{d^2 + 4c_{xy}^2}}{2c_{xy}} = a \tag{19}$$

$$\cos\delta = \frac{1}{\sqrt{1+a^2}} \qquad \sin\delta = \frac{a}{\sqrt{1+a^2}} \tag{20}$$

$$\tan(2\delta) = \frac{-2c_{xy}}{d} = \frac{2a}{1-a^2} \tag{21}$$

$$\cos(2\delta) = \frac{-d}{\sqrt{d^2+4c_{xy}^2}} = \frac{1-a^2}{1+a^2} \qquad \sin(2\delta) = \frac{2c_{xy}}{\sqrt{d^2+4c_{xy}^2}} = \frac{2a}{1+a^2} \tag{22}$$

As can be seen from (11), the x' axis is pointing in the direction $(\cos\delta,\ \sin\delta) = \frac{1}{\sqrt{1+a^2}}(1,\ a) = \vec{a}$, the y' axis in the direction $(-\sin\delta,\ \cos\delta) = \frac{1}{\sqrt{1+a^2}}(-a,\ 1) = \vec{a}_T$. In the original camera system the shower axis can be described by

$$(y - y_0) = a\cdot(x - x_0) + b \tag{23}$$

with

$$b = (\overline{y} - y_0) - a\cdot(\overline{x} - x_0) \tag{24}$$

The expressions for $c_{x'x'}$, $c_{x'y'}$ and $c_{y'y'}$ read

$$\begin{aligned}
c_{x'x'} &= \cos^2\delta\cdot c_{xx} + 2\sin\delta\cos\delta\cdot c_{xy} + \sin^2\delta\cdot c_{yy} \\
&= \frac{1+\cos(2\delta)}{2}\cdot c_{xx} + \sin(2\delta)\cdot c_{xy} + \frac{1-\cos(2\delta)}{2}\cdot c_{yy} \\
&= \big(c_{xx} + 2a\cdot c_{xy} + a^2\cdot c_{yy}\big)\ /\ \big(1+a^2\big) \\
c_{x'y'} &= 0 \\
c_{y'y'} &= \sin^2\delta\cdot c_{xx} - 2\sin\delta\cos\delta\cdot c_{xy} + \cos^2\delta\cdot c_{yy} \\
&= \frac{1-\cos(2\delta)}{2}\cdot c_{xx} - \sin(2\delta)\cdot c_{xy} + \frac{1+\cos(2\delta)}{2}\cdot c_{yy} \\
&= \big(a^2\cdot c_{xx} - 2a\cdot c_{xy} + c_{yy}\big)\ /\ \big(1+a^2\big)
\end{aligned} \tag{25}$$

Because

$$\begin{aligned}
c_{x'x'} - c_{y'y'} &= \cos^2\delta\cdot(c_{xx}-c_{yy}) - \sin^2\delta\cdot(c_{xx}-c_{yy}) + 2\sin(2\delta)\cdot c_{xy} \\
&= \cos(2\delta)\cdot(c_{xx}-c_{yy}) + 2\sin(2\delta)\cdot c_{xy} \\
&= \sqrt{d^2+4c_{xy}^2} \ \geq\ 0
\end{aligned} \tag{26}$$

the x-axis of the rotated system is along the main axis of the shower image.

If one had chosen in eq.(18) the $-$ sign instead of the $+$ sign, (2δ) would change by $180°$ and the expressions for $\cos(2\delta)$ and $\sin(2\delta)$ would be

$$\cos(2\delta) \;=\; \frac{d}{\sqrt{d^2+4c_{xy}^2}} \;=\; \frac{1-a^2}{1+a^2} \qquad\qquad \sin(2\delta) \;=\; \frac{-2c_{xy}}{\sqrt{d^2+4c_{xy}^2}} \;=\; \frac{2a}{1+a^2} \tag{27}$$

This would imply

$$\begin{aligned} c_{x'x'} - c_{y'y'} &= \cos^2\delta\cdot(c_{xx}-c_{yy}) - \sin^2\delta\cdot(c_{xx}-c_{yy}) + 2\sin(2\delta)\cdot c_{xy} \\ &= \cos(2\delta)\cdot(c_{xx}-c_{yy}) + 2\sin(2\delta)\cdot c_{xy} \\ &= -\sqrt{d^2+4c_{xy}^2} \;\le\; 0 \end{aligned} \tag{28}$$

which means that the x-axis of the rotated system would be along the smaller axis of the shower image.

4 Definition of the image parameters

The image parameters (see also Fig. 1) can be grouped into three classes :

- those describing the shape of the shower and which are therefore independent of a reference direction and independent of a reference point,
- those depending on a reference direction, and
- those depending on a reference point

Shape parameters

SIZE	total number of photons in the shower
LENGTH	half length of the shower image
WIDTH	half width of the shower image
ASYM	asymmetry of the photon distribution along the shower axis
MDIST	distance between the center of the shower image and the brightest pixel
CONC	fraction of photons contained in the 2 brightest pixels
LEAKAGE	fraction of photons contained in the pixels close to the edge of the camera

Parameter depending on a reference direction

DELTA	angle δ between the shower axis and the x-axis of the camera (independent of head-tail information)

Parameters depending on the reference point (RP)

DIST	distance of center of shower image from the RP
MISS	distance of the shower axis from the RP
ALPHA	angle α by which shower axis misses the RP, as seen from the center of the shower image
BETA	angle β between the x-axis of the camera and the line connecting the RP with the center of the shower image
AZWIDTH	half width of the shower image as seen from the RP

Using the formulas of the previous section the image parameters can be written as (see Fig. 1)

$$\begin{aligned}
\mathrm{LENGTH}^2 &= c_{x'x'} && (29)\\
&= \cos^2\delta\cdot c_{xx} \;+\; 2\sin\delta\cos\delta\cdot c_{xy} \;+\; \sin^2\delta\cdot c_{yy} && (30)\\
&= \frac{1+\cos(2\delta)}{2}\cdot c_{xx} \;+\; \sin(2\delta)\cdot c_{xy} \;+\; \frac{1-\cos(2\delta)}{2}\cdot c_{yy} && (31)\\
&= \left(c_{xx} \;+\; 2a\cdot c_{xy} \;+\; a^2\cdot c_{yy}\right) \;/\; \left(1+a^2\right) && (32)\\
\mathrm{WIDTH}^2 &= c_{y'y'} && (33)\\
&= \sin^2\delta\cdot c_{xx} \;-\; 2\sin\delta\cos\delta\cdot c_{xy} \;+\; \cos^2\delta\cdot c_{yy} && (34)\\
&= \frac{1-\cos(2\delta)}{2}\cdot c_{xx} \;-\; \sin(2\delta)\cdot c_{xy} \;+\; \frac{1+\cos(2\delta)}{2}\cdot c_{yy} && (35)\\
&= \left(a^2\cdot c_{xx} \;-\; 2a\cdot c_{xy} \;+\; c_{yy}\right) \;/\; \left(1+a^2\right) && (36)\\
\mathrm{ASYM}^3 &= c_{x'^3} && (37)\\
&= \cos^3\delta\cdot c_{x^3} \;+\; 3\cos^2\delta\sin\delta\cdot c_{x^2y} \;+\; 3\cos\delta\sin^2\delta\cdot c_{xy^2} \;+\; \sin^3\delta\cdot c_{y^3} && (38)
\end{aligned}$$

$$\begin{aligned}
\mathrm{DIST}^2 &= (\overline{x}-x_0)^2 \;+\; (\overline{y}-y_0)^2 && (39)\\
\mathrm{MISS} &= b\cdot\cos\delta \;=\; \frac{b}{\sqrt{1+a^2}} && (40)
\end{aligned}$$

The angle BETA ($=\beta$) is obtained from

$$\sin\beta \;=\; (\overline{y}-y_0)\;/\;\mathrm{DIST} \qquad \cos\beta \;=\; (\overline{x}-x_0)\;/\;\mathrm{DIST} \tag{41}$$

with $-180^\circ \;<\; \beta \;<\; 180^\circ$.
The RMS^2 of the photon distribution along a direction given by the unit vector $\vec{e} = (e_x, e_y)$ is calculated as

$$c_{\vec{e}} \;=\; (e_x \;\; e_y)\cdot\begin{pmatrix} c_{xx} & c_{xy}\\ c_{xy} & c_{yy}\end{pmatrix}\cdot\begin{pmatrix} e_x\\ e_y\end{pmatrix} \tag{42}$$

$\mathrm{AZWIDTH}^2$ is the RMS^2 along the direction

$$\vec{e} \;=\; \frac{1}{\sqrt{(\overline{x}-x_0)^2+(\overline{y}-y_0)^2}}\begin{pmatrix} -(\overline{y}-y_0)\\ (\overline{x}-x_0)\end{pmatrix} \tag{43}$$

which is perpendicular to the vector $\vec{d} = (\overline{x}-x_0, \overline{y}-y_0)$. Inserting $\vec{e}$ into (42) yields

$$\mathrm{AZWIDTH}^2 \;=\; \frac{(\overline{y}-y_0)^2\cdot c_{xx} - 2(\overline{x}-x_0)\cdot(\overline{y}-y_0)\cdot c_{xy} + (\overline{x}-x_0)^2\cdot c_{yy}}{(\overline{x}-x_0)^2+(\overline{y}-y_0)^2} \tag{44}$$

5 The ALPHA parameter (α)

5.1 General definition of α

The ALPHA parameter (α) describes the orientation of the shower image relative to a reference point (x_0, y_0). It tells by which angle the main axis of the image misses the reference point. In the definition of ALPHA the head-tail information may be incorporated.

The α parameter involves the three vectors $\vec{a}$, $\vec{d}$ and $\vec{c}$ (see Fig. 2), where

$$\vec{a} \;=\; \frac{1}{\sqrt{1+a^2}} \begin{pmatrix} 1 \\ a \end{pmatrix} \tag{45}$$

is a vector along the main axis of the shower image (which according to the definition in Sect. 3 does not contain the head-tail information),

$$\vec{d} \;=\; \begin{pmatrix} \overline{x} - x_0 \\ \overline{y} - y_0 \end{pmatrix} \tag{46}$$

is the distance vector from the reference point (x_0, y_0) to the center $(\overline{x}, \overline{y})$ of the shower image.

$\vec{c}$ is a vector containing the head-tail information, such that $(\vec{c}\cdot\vec{a})\;\vec{a}$ is a vector which is pointing from the head to the tail of the shower and which is parallel (or antiparallel) to $\vec{a}$. Due to fluctuations in the shower development and because of experimental errors the vector $\vec{c}$ is not necessarily parallel (or antiparallel) to $\vec{a}$. The head-tail information can be obtained in several ways. For example

- from the position $(x_P,\ y_P)$ of the maximum of the distribution of the number N_i of photons : $\vec{c} \;=\; \begin{pmatrix} \overline{x} - x_P \\ \overline{y} - y_P \end{pmatrix}$
- from the third moment ASYM[3] of the N_i distribution along the x' axis : $(\vec{c}\cdot\vec{a}) \;= \mathrm{ASYM}$
- from the arrival times of the photons in the different pixels, the photons from the head of the shower arriving earlier than those from the tail. Also this information can be transformed into a vector $\vec{c}$ pointing roughly from the head to the tail.

With the notation

$$s \;=\; \mathrm{sign}(\vec{c}\cdot\vec{a}) \tag{47}$$

the image parameter ALPHA ($=\alpha$) is defined by

$$\cos\alpha = \; s\cdot\frac{(\vec{a}\cdot\vec{d})}{|\vec{a}||\vec{d}|} \;=\; s\cdot\frac{(\overline{x}-x_0)+a(\overline{y}-y_0)}{\mathrm{DIST}\cdot\sqrt{1+a^2}} \tag{48}$$

$$\sin\alpha \;=\; s\cdot\frac{(\vec{a}\times\vec{d})_z}{|\vec{a}||\vec{d}|} \;=\; s\cdot\frac{(\overline{y}-y_0)-a(\overline{x}-x_0)}{\mathrm{DIST}\cdot\sqrt{1+a^2}} \tag{49}$$

$$\;=\; \frac{s\cdot b}{\mathrm{DIST}\cdot\sqrt{1+a^2}} \tag{50}$$

with $-180^\circ < \alpha < 180^\circ$.

In Fig. 3 (upper plot) the α angles are drawn for different showers which have their centers at the edge of the camera. In all cases it is assumed that the main axis of the showers is horizonthal, that the head-tail vector is pointing to the right and that the source position (x_0, y_0) coincides with the center of the camera $(\overline{x}, \overline{y})$. For a rotational symmetric camera, which for MAGIC is approximately the case, it is sufficient to consider only this specific orientation of the showers.

The dashed lines connect centers of showers which have the same α. For example the small circles in Fig. 3 represent centers of showers which all have $\alpha = \alpha_1$.

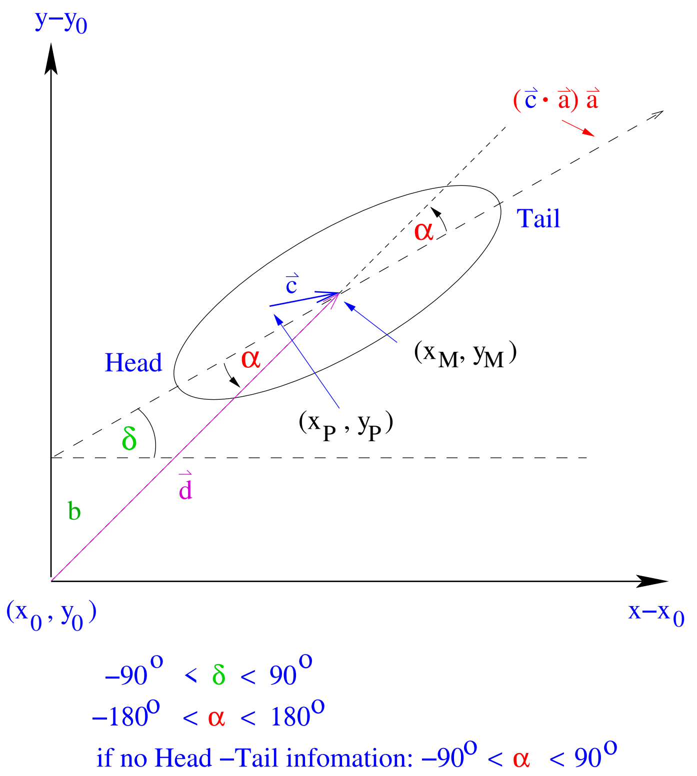

Figure 2: Shower image with head-tail information. $\vec{c}$ is a vector pointing roughly from the head to the tail of the shower, $\vec{a}$ is a vector along the main axis of the shower, and $(\vec{c}\cdot\vec{a})\ \vec{a}$ is a vector which is pointing from the head to the tail of the shower and which is parallel (or antiparallel) to $\vec{a}$. The center of the shower $(\overline{x}, \overline{y})$ is denoted as (x_M, y_M). Due to fluctuations in the shower development and because of measurement errors $\vec{c}$ is not necessarily parallel (or antiparallel) to $\vec{a}$.

The shower images of photons coming from the source are expected to have small $|\alpha|$, because the shower axis should go through the source position and the head-tail vector should point away from the source position. Because of measurement errors the head-tail direction may be wrong leading to some α values around 180° (see lower plot of Fig. 3). The shower images of showers from the isotropic hadronic background on the other hand populate the whole α range, with some anisotropy due to different reconstruction efficiences for different positions of the showers in the camera.

5.2 Separate treatment of the head-tail information

In practice, often the head-tail information (and thus the sign s) is not available. Therefore and also to avoid the multi-peak structure of the photon signal (Fig. 3) it is useful to define α with $s = 1$, and to use the head-tail information separately :

$$\begin{aligned}\sin\alpha &= \frac{(\vec{a}\times\vec{d})_z}{|\vec{a}||\vec{d}|} = \frac{(\overline{y}-y_0) - a(\overline{x}-x_0)}{\mathrm{DIST}\cdot\sqrt{1+a^2}} \\ &= \frac{b}{\mathrm{DIST}\cdot\sqrt{1+a^2}} = \frac{\mathrm{MISS}}{\mathrm{DIST}}\end{aligned} \qquad (51)$$

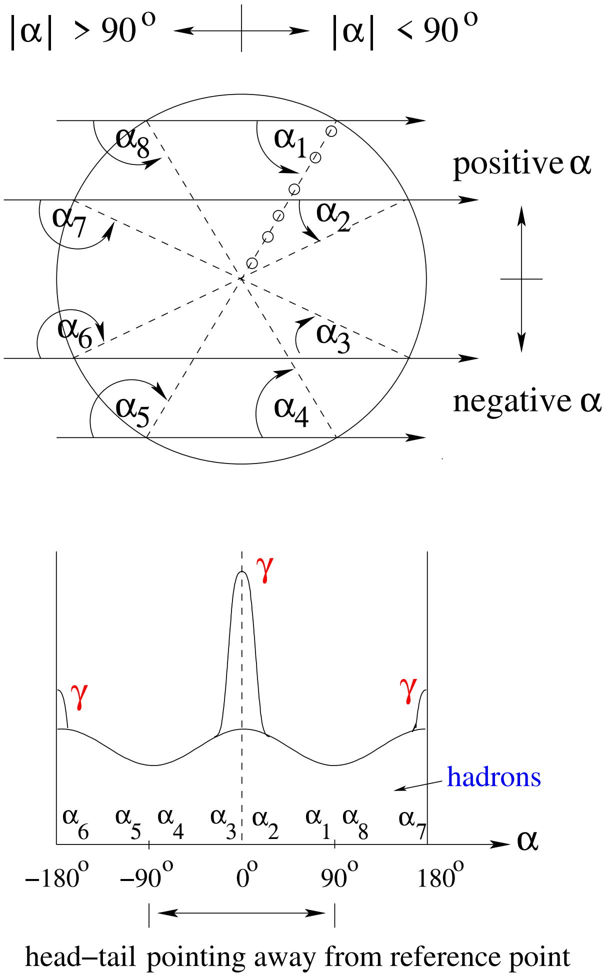

Figure 3: Illustration of the different values of α (=ALPHA) for showers images whose main axis is horizonthal and whose head-tail vector is pointing to the right. The big circle represents the contour of the camera and the dotted lines connect centers of shower images which have the same α. Lower plot : Expected experimental distributions of α for photons from a point source (γ) and for the isotropic hadronic cosmic ray background (*hadrons*). For calculating α the source position, which is assumed to be the center of the camera, is taken as the reference point (x_0, y_0).

with $-90^\circ < \alpha < 90^\circ$. This new α (defined by (51)) is obtained from the α_{360} (defined by (48) and (49)) through

$$\alpha = \begin{cases} \alpha_{360} & \text{if} \quad -90^\circ < \alpha_{360} < 90^\circ \\ \alpha_{360} + 180^\circ & \text{if} \quad -180^\circ < \alpha_{360} < -90^\circ \\ \alpha_{360} - 180^\circ & \text{if} \quad 90^\circ < \alpha_{360} < 180^\circ \end{cases} \tag{52}$$

or

$$\alpha = \mathrm{MOD}(\alpha_{360} + 270^\circ,\ 180^\circ) - 90^\circ \tag{53}$$

The head-tail information, if available, can be taken into account by considering the sign of $\cos\alpha$ (eq.(48)) together with its error. The sign of $\cos\alpha$ is equal to the sign of the quantity

$$t = (\vec{c} \cdot \vec{a}) \cdot (\vec{a} \cdot \vec{d}) \tag{54}$$

The experimental error of t involves the errors of $\vec{c}$, $\vec{a}$ and $\vec{d}$, which makes the error calculation complicated. Since in general only those showers are of interest for which $\vec{a}$ and $\vec{d}$, and thus also $(\vec{a} \cdot \vec{d})$, are reasonably well measured it will be sufficient to define t as

$$t = (\vec{c} \cdot \vec{a}) \cdot \mathrm{sign}(\vec{a} \cdot \vec{d}) \tag{55}$$

and to consider only the error of the asymmetry $(\vec{c} \cdot \vec{a})$ along the main axis $\vec{a}$ of the shower image. If the asymmetry is given by ASYM, t and its error become

$$t = \mathrm{ASYM} \cdot \mathrm{sign}(\vec{a} \cdot \vec{d}) \tag{56}$$

$$\Delta t = \Delta\mathrm{ASYM} \tag{57}$$

A selection

$$t > -n \cdot \Delta t \tag{58}$$

(where n is a number in the order of 1) will keep only those showers which are consistent with a positive sign of t. The selection will thus keep nearly all photons from the source and will reject essentially only hadrons. Monte Carlo calculations have to be performed to check whether this approach is reasonable.

6 Errors of the image parameters

6.1 Error of the rotation angle δ

The rotation angle δ (eq.(19)) is a function of $(q_1,\ q_2,\ q_3) = (c_{xx},\ c_{yy},\ c_{xy})$, which in turn are functions of w_k (see (2), (3) and (4)), which in turn are functions of N_l (see (1)). The experimental error of δ results from the experimental error of N_l :

$$\Delta\delta\Delta\delta = \left(\sum_l \frac{\partial\delta}{\partial N_l}\cdot\Delta N_l\right)^2 = \sum_{jl}\frac{\partial\delta}{\partial N_j}\cdot\frac{\partial\delta}{\partial N_l}\cdot(\Delta N_j\Delta N_l) \tag{59}$$

$$\frac{\partial\delta}{\partial N_l} = \sum_i \frac{\partial\delta}{\partial q_i}\cdot\frac{\partial q_i}{\partial N_l} \tag{60}$$

$$\frac{\partial\delta}{\partial q_1} = \frac{\partial\delta}{\partial c_{xx}} = \frac{-c_{xy}}{d^2+4c_{xy}^2} \tag{61}$$

$$\frac{\partial\delta}{\partial q_2} = \frac{\partial\delta}{\partial c_{yy}} = \frac{c_{xy}}{d^2+4c_{xy}^2} \tag{62}$$

$$\frac{\partial\delta}{\partial q_3} = \frac{\partial\delta}{\partial c_{xy}} = \frac{-d}{d^2+4c_{xy}^2} \tag{63}$$

$$\frac{\partial w_k}{\partial N_l} = \frac{\delta_{kl}}{\sum_j N_j} - \frac{N_k}{\left(\sum_j N_j\right)^2} = \frac{1}{\sum_j N_j}\cdot\left(\delta_{kl} - \frac{N_k}{\sum_j N_j}\right) = \frac{1}{\sum_j N_j}\cdot(\delta_{kl} - w_k) \tag{64}$$

$$\frac{\partial q_i}{\partial N_l} = \sum_k \frac{\partial q_i}{\partial w_k}\cdot\frac{\partial w_k}{\partial N_l} \tag{65}$$

$$\frac{\partial q_1}{\partial w_k} = \frac{\partial c_{xx}}{\partial w_k} = x_k^2 - 2\overline{x}x_k = (x_k-\overline{x})^2 - \overline{x}^2 \tag{66}$$

$$\frac{\partial q_2}{\partial w_k} = \frac{\partial c_{yy}}{\partial w_k} = y_k^2 - 2\overline{y}y_k = (y_k-\overline{y})^2 - \overline{y}^2 \tag{67}$$

$$\frac{\partial q_3}{\partial w_k} = \frac{\partial c_{xy}}{\partial w_k} = x_k y_k - \overline{y}x_k - \overline{x}y_k = (x_k-\overline{x})(y_k-\overline{y}) - \overline{x}\,\overline{y} \tag{68}$$

$$\frac{\partial q_1}{\partial N_l} = \sum_k \frac{\partial q_1}{\partial w_k}\cdot\frac{\partial w_k}{\partial N_l} = \frac{1}{\sum_j N_j}\cdot\left\{(x_l-\overline{x})^2 - c_{xx}\right\} \tag{69}$$

$$\frac{\partial q_2}{\partial N_l} = \sum_k \frac{\partial q_2}{\partial w_k}\cdot\frac{\partial w_k}{\partial N_l} = \frac{1}{\sum_j N_j}\cdot\left\{(y_l-\overline{y})^2 - c_{yy}\right\} \tag{70}$$

$$\frac{\partial q_3}{\partial N_l} = \sum_k \frac{\partial q_3}{\partial w_k}\cdot\frac{\partial w_k}{\partial N_l} = \frac{1}{\sum_j N_j}\cdot\{(x_l-\overline{x})(y_l-\overline{y}) - c_{xy}\} \tag{71}$$

$$\frac{\partial\delta}{\partial N_l} = \sum_i \frac{\partial\delta}{\partial q_i}\cdot\frac{\partial q_i}{\partial N_l} \tag{72}$$

$$= \frac{1}{d^2+4c_{xy}^2}\cdot\frac{1}{\sum_j N_j}\cdot\left\{-d(x_l-\overline{x})(y_l-\overline{y}) + c_{xy}\left[(y_l-\overline{y})^2 - (x_l-\overline{x})^2\right]\right\} \tag{73}$$

Inserting (73) into (59) the error of δ can be calculated from the error matrix $(\Delta N_j \Delta N_l)$. If this error matrix is diagonal, which means that the errors of N_j and of N_l are uncorrelated, (59) reduces to

$$\Delta\delta\Delta\delta = \sum_j \left(\frac{\partial\delta}{\partial N_j}\right)^2 (\Delta N_j)^2 \tag{74}$$

6.2 Error of the image parameter ALPHA (α)

The error calculation of α is done using the expression for $\tan\alpha$

$$\tan\alpha = \frac{\cos\delta(\overline{y}-y_0) - \sin\delta(\overline{x}-x_0)}{\sin\delta(\overline{y}-y_0) + \cos\delta(\overline{x}-x_0)} \tag{75}$$

which follows from eqs. (48) and (49). α is considered as a function of δ and, via $\overline{x}$ and $\overline{y}$, of w_k. In turn, δ and w_k depend on N_l. The experimental error of α results from the experimental error of N_l :

$$\Delta\alpha\Delta\alpha = \left(\sum_l \frac{\partial\alpha}{\partial N_l}\cdot\Delta N_l\right)^2 = \sum_{jl} \frac{\partial\alpha}{\partial N_j}\cdot\frac{\partial\alpha}{\partial N_l}\cdot(\Delta N_j \Delta N_l) \tag{76}$$

$$\frac{\partial\alpha}{\partial N_l} = \frac{\partial\alpha}{\partial\delta}\cdot\frac{\partial\delta}{\partial N_l} + \sum_k \frac{\partial\alpha}{\partial w_k}\cdot\frac{\partial w_k}{\partial N_l} \tag{77}$$

$$\frac{\partial\alpha}{\partial\delta} = -1 \tag{78}$$

$$\frac{\partial\alpha}{\partial w_k} = \frac{1}{\mathrm{DIST}^2}\cdot\{y_k(\overline{x}-x_0) \;-\; x_k(\overline{y}-y_0)\} \tag{79}$$

$$\sum_k \frac{\partial\alpha}{\partial w_k}\cdot\frac{\partial w_k}{\partial N_l} = \frac{1}{\sum_j N_j}\cdot\frac{(x_l-\overline{x})(y_0-\overline{y}) \;-\; (y_l-\overline{y})(x_0-\overline{x})}{\mathrm{DIST}^2} \tag{80}$$

$$\frac{\partial\alpha}{\partial N_l} = -\frac{\partial\delta}{\partial N_l} + \frac{1}{\sum_j N_j}\cdot\frac{(x_l-\overline{x})(y_0-\overline{y}) \;-\; (y_l-\overline{y})(x_0-\overline{x})}{\mathrm{DIST}^2} \tag{81}$$

6.3 Errors of LENGTH and WIDTH

Both LENGTH^2 and WIDTH^2 are denoted by the same expression L^2 in which the upper sign refers to LENGTH^2 and the lower sign to WIDTH^2 :

$$\mathrm{L}^2 = c_{xx}\cdot\frac{1\pm\cos 2\delta}{2} \;\pm\; \sin 2\delta\cdot c_{xy} \;+\; c_{yy}\cdot\frac{1\mp\cos 2\delta}{2} \tag{82}$$

L^2 is considered as a function of δ and $(q_1,\ q_2,\ q_3) \;=\; (c_{xx}, c_{yy}, c_{xy})$, which in turn are functions of N_l. The error of L results from the experimental error of N_l.

$$\Delta \mathrm{L}^2 \Delta \mathrm{L}^2 = \left(\sum_l \frac{\partial \mathrm{L}^2}{\partial N_l} \cdot \Delta N_l \right)^2 = \sum_{jl} \frac{\partial \mathrm{L}^2}{\partial N_j} \cdot \frac{\partial \mathrm{L}^2}{\partial N_l} \cdot (\Delta N_j \Delta N_l) \tag{83}$$

$$\frac{\partial \mathrm{L}^2}{\partial N_l} = \frac{\partial \mathrm{L}^2}{\partial \delta} \cdot \frac{\partial \delta}{\partial N_l} + \sum_i \frac{\partial \mathrm{L}^2}{\partial q_i} \cdot \frac{\partial q_i}{\partial N_l} \tag{84}$$

$$\frac{\partial \mathrm{L}^2}{\partial \delta} = \mp \sin(2\delta) \cdot (c_{xx} - c_{yy}) \pm 2\cos(2\delta) \cdot c_{xy} \tag{85}$$

$$\frac{\partial \mathrm{L}^2}{\partial q_1} = \frac{\partial \mathrm{L}^2}{\partial c_{xx}} = \frac{1 \pm \cos(2\delta)}{2} \tag{86}$$

$$\frac{\partial \mathrm{L}^2}{\partial q_2} = \frac{\partial \mathrm{L}^2}{\partial c_{yy}} = \frac{1 \mp \cos(2\delta)}{2} \tag{87}$$

$$\frac{\partial \mathrm{L}^2}{\partial q_3} = \frac{\partial \mathrm{L}^2}{\partial c_{xy}} = \pm \sin(2\delta) \tag{88}$$

6.4 Error of the asymmetry ASYM

ASYM (=A) (eq.(38)) is considered as a function of δ and $(s_1,\ s_2,\ s_3,\ s_4) = (c_{x^3},\ c_{y^3},\ c_{xy^2},\ c_{x^2y})$, which in turn are functions of N_l. The error of A results from the experimental error of N_l.

$$\Delta A \Delta A = \left(\sum_l \frac{\partial A}{\partial N_l} \cdot \Delta N_l \right)^2 = \sum_{jl} \frac{\partial A}{\partial N_j} \cdot \frac{\partial A}{\partial N_l} \cdot (\Delta N_j \Delta N_l) \tag{89}$$

$$\frac{\partial \mathrm{A}}{\partial N_l} = \frac{d\mathrm{A}}{d\mathrm{A}^3} \cdot \left(\frac{\partial \mathrm{A}^3}{\partial \delta} \cdot \frac{\partial \delta}{\partial N_l} + \sum_i \frac{\partial \mathrm{A}^3}{\partial s_i} \cdot \frac{\partial s_i}{\partial N_l} \right) \tag{90}$$

$$\frac{dA}{dA^3} = \frac{1}{3A^2} \tag{91}$$

$$\begin{aligned} \frac{\partial \mathrm{A}^3}{\partial \delta} = & -3c_{x^3} \cdot \cos^2\delta \sin\delta + 3c_{x^2y} \cdot \cos\delta \cdot \left(1 - 3\sin^2\delta\right) \\ & -3c_{xy^2} \cdot \sin\delta \cdot \left(1 - 3\cos^2\delta\right) + 3c_{y^3} \cdot \sin^2\delta \cos\delta \end{aligned} \tag{92}$$

$$\frac{\partial \mathrm{A}^3}{\partial s_1} = \frac{\partial \mathrm{A}^3}{\partial c_{x^3}} \ = \ \cos^3\delta \tag{93}$$

$$\frac{\partial \mathrm{A}^3}{\partial s_2} = \frac{\partial \mathrm{A}^3}{\partial c_{y^3}} \ = \ \sin^3\delta \tag{94}$$

$$\frac{\partial \mathrm{A}^3}{\partial s_3} = \frac{\partial \mathrm{A}^3}{\partial c_{xy^2}} \ = \ 3\cos\delta\sin^2\delta \tag{95}$$

$$\frac{\partial \mathrm{A}^3}{\partial s_4} = \frac{\partial \mathrm{A}^3}{\partial c_{x^2y}} \ = \ 3\cos^2\delta\sin\delta \tag{96}$$

$$\frac{\partial s_1}{\partial w_k} = (x_k - \overline{x})^3 \ - \ 3x_k \cdot c_{xx} \ + \ \overline{x}^3 \tag{97}$$

$$\frac{\partial s_2}{\partial w_k} = (y_k - \overline{y})^3 \ - \ 3y_k \cdot c_{yy} \ + \ \overline{y}^3 \tag{98}$$

$$\frac{\partial s_3}{\partial w_k} = (x_k - \overline{x})(y_k - \overline{y})^2 \ - \ x_k \cdot c_{yy} \ - \ 2y_k \cdot c_{xy} \ + \ \overline{x}\,\overline{y}^2 \tag{99}$$

$$\frac{\partial s_4}{\partial w_k} = (x_k - \overline{x})^2(y_k - \overline{y}) \ - \ y_k \cdot c_{xx} \ - \ 2x_k \cdot c_{xy} \ + \ \overline{x}^2\,\overline{y} \tag{100}$$

$$\begin{aligned}\frac{\partial s_1}{\partial N_l} &= \frac{\partial s_1}{\partial w_k} \cdot \frac{\partial w_k}{\partial N_l} \\ &= \frac{1}{\sum_j N_j} \cdot \left\{(x_l - \overline{x})^3 \ - \ 3(x_l - \overline{x}) \cdot c_{xx} \ - \ c_{x^3}\right\}\end{aligned} \tag{101}$$

$$\begin{aligned}\frac{\partial s_2}{\partial N_l} &= \frac{\partial s_2}{\partial w_k} \cdot \frac{\partial w_k}{\partial N_l} \\ &= \frac{1}{\sum_j N_j} \cdot \left\{(y_l - \overline{y})^3 \ - \ 3(y_l - \overline{y}) \cdot c_{yy} \ - \ c_{y^3}\right\}\end{aligned} \tag{102}$$

$$\begin{aligned}\frac{\partial s_3}{\partial N_l} &= \frac{\partial s_3}{\partial w_k} \cdot \frac{\partial w_k}{\partial N_l} \\ &= \frac{1}{\sum_j N_j} \cdot \left\{(x_l - \overline{x})(y_l - \overline{y})^2 \ - \ (x_l - \overline{x}) \cdot c_{yy} \ - \ 2(y_l - \overline{y}) \cdot c_{xy} \ - \ c_{xy^2}\right\}\end{aligned} \tag{103}$$

$$\begin{aligned}\frac{\partial s_4}{\partial N_l} &= \frac{\partial s_4}{\partial w_k} \cdot \frac{\partial w_k}{\partial N_l} \\ &= \frac{1}{\sum_j N_j} \cdot \left\{(x_l - \overline{x})^2(y_l - \overline{y}) \ - \ (y_l - \overline{y}) \cdot c_{xx} \ - \ 2(x_l - \overline{x}) \cdot c_{xy} \ - \ c_{x^2y}\right\}\end{aligned} \tag{104}$$

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